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1994 J. Phys. A: Math. Gen. 27 2083

(http://iopscience.iop.org/0305-4470/27/6/031)

J. Phys. A: Math. Gen. 27 (1994) 2083–2092. Printed in the UK

On the construction of approximate solutions for a multidimensional nonlinear heat equation

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Received 25 August 1993

Abstract. We study three methods, based on continuous symmetries, to find approximate solutions for the multidimensional nonlinear heat equation $\partial u/\partial x_0 + \Delta u = au^n + \varepsilon f(u)$, where a and n are arbitrary real constants, f is a smooth function, and $0 < \varepsilon << 1$.

1. Introduction

Recently, we studied approximate symmetries for a Landau–Ginzburg equation (Euler *et al* 1992). Within this approach one can obtain approximate solutions for multidimensional partial differential equations with a small parameter (Shul'ga 1987, Fushchich and Shtelen 1989). Moreover, Baikov *et al* (1989) introduced a different definition of approximate symmetries in order to obtain exact solutions for such equations. In this paper we attempt to develop the method of approximate solutions in a study of the following multidimensional nonlinear heat equation

$$\frac{\partial u}{\partial x_0} + \Delta u = au^n + \varepsilon f(u) \tag{1}$$

where a and n are real constants, $0 < \varepsilon \ll 1$, f is a smooth function, and $\Delta = \sum_{j=1}^{3} \partial^2 u/\partial x_j^2$. A classification of f in (1) for an approximate scaling symmetry and an approximate Lie–Bäcklund symmetry is performed. A more general concept of approximation together with its compatibility problem is also studied. Approximate solutions are calculated. Finally we consider the method of Baikov *et al* (1989) for approximate scaling invariance.

2. *Approximate scaling invariance*

Let us first consider the concept of approximate systems for the construction of approximate solutions. Here an approximate system is obtained by representing the solution u of a (nonlinear) partial differential equation in the form

$$u = u_0 + \varepsilon u_1 + O(\varepsilon^2) \tag{2}$$

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where u_j ($j = 0, 1, \ldots$) are smooth functions of the independent variables. After substituting and equating to zero the coefficients of zero and different powers of ε, we obtain systems of partial differential equations.

With the representation (2), a first-order approximation of (1) is given by the following system of partial differential equations:

$$\begin{aligned}\frac{\partial u_0}{\partial x_0} + \Delta u_0 &= a u_0^n \\ \frac{\partial u_1}{\partial x_0} + \Delta u_1 &= a n u_0^{n-1} u_1 + f(u_0).\end{aligned} \tag{3}$$

Equation (1) and system (3) admit the translation and rotation symmetry vector fields

$$\frac{\partial}{\partial x_i} \qquad \text{and} \qquad x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j}$$

where $i = 0, \ldots, 3$ and $j \neq k = 1, \ldots, 3$.

We consider the following scaling symmetry generator for system (3):

$$Z = 2c_1 x_0 \frac{\partial}{\partial x_0} + c_1 \sum_{i=1}^{3} x_i \frac{\partial}{\partial x_i} + c_2 u_0 \frac{\partial}{\partial u_0} + \eta_1(u_0, u_1) \frac{\partial}{\partial u_1} \tag{4}$$

where η_1 is an arbitrary function of its arguments. For

$$n = 1 - \frac{2c_1}{c_2} \tag{5}$$

($c_2 \neq 0$) it follows that η_1 must take the form

$$\eta_1(u_0, u_1) = k_1 u_0 + k_2 u_1 + k_3 \tag{6}$$

where $c_1, c_2, k_1, k_2, k_3 \in \mathcal{R}$.

Theorem 1. The function f in (1), such that Z is a first-order approximate symmetry for (1), is given by the following three cases:

(i) For $k_2 \neq 0$ and $k_2 \neq c_2$ we obtain

$$f(u) = \frac{2ac_1k_1}{c_2(c_2 - k_2)} u^{1-2c_1/c_2} + \left(1 - \frac{2c_1}{c_2}\right) \frac{ak_3}{k_2} u^{-2c_1/c_2} + c u^{(k_2-2c_1)/c_2}. \tag{7}$$

(ii) For $k_2 = 0$ we obtain

$$f(u) = \frac{2ac_1k_1}{c_2^2} u^{1-2c_1/c_2} - \left(1 - \frac{2c_1}{c_2}\right) \frac{ak_3}{c_2} u^{-2c_1/c_2} \ln|u| + c u^{-2c_1/c_2}. \tag{8}$$

(iii) For $k_2 = c_2$ we obtain

$$f(u) = \frac{ac_1k_1}{c_2(c_2 - 2c_1)} u^{1-2c_1/c_2} - \frac{ak_3(c_2 - 2c_1)}{c_2(c_2 - 4c_1)} u^{-2c_1/c_2} + c u^{-(1-2c_1/c_2)}. \tag{9}$$

In the above given cases $c \in \mathcal{R}$.

To prove this theorem we make use of the Lie derivative for the invariance condition of system (3) with respect to the Lie symmetry (4) whereby the functions (7), (8) and (9) follow. For more details on Lie symmetry calculations we refer the reader to Ovsiannikov (1982), Olver (1986), Bluman and Kumei (1989), Fushchich *et al* (1993), and Euler and Steeb (1992).

We now construct a symmetry ansatz from the scaling symmetry for the function in case (i), i.e. we have to solve the Lagrange system

$$\frac{\mathrm{d}x_0}{2c_1x_0} = \frac{\mathrm{d}x_1}{c_1x_1} = \frac{\mathrm{d}x_2}{c_1x_2} = \frac{\mathrm{d}x_3}{c_1x_3} = \frac{\mathrm{d}u_0}{c_2u_0} = \frac{\mathrm{d}u_1}{k_1u_0+k_2u_1+k_3} := \mathrm{d}\tau$$

where τ is a group parameter. The following ansatz is obtained:

$$\begin{aligned} u_0 &= x_0^{1/(1-n)}\varphi_0(\omega_1,\omega_2,\omega_3) \\ u_1 &= \frac{k_1}{c_2-k_2}x_0^{1/(1-n)}\varphi_0(\omega_1,\omega_2,\omega_3) + x_0^{k_2/(2c_1)}\varphi_1(\omega_1,\omega_2,\omega_3) - \frac{k_3}{k_2} \end{aligned} \tag{10}$$

where

$$\omega_1 = \frac{x_0}{x_1^2} \qquad \omega_2 = \frac{x_1}{x_2} \qquad \omega = \frac{x_2}{x_3}.$$

System (3) reduces to

$$\begin{aligned} 4\omega_1^3\frac{\partial^2\varphi_0}{\partial\omega_1^2} &+ \omega_1\omega_2^2(1+\omega_2^2)\frac{\partial^2\varphi_0}{\partial\omega_2^2} + \omega_1\omega_2^2\omega_3^2(1+\omega_3^2)\frac{\partial^2\varphi_0}{\partial\omega_3^2} - 4\omega_1^2\omega_2\frac{\partial^2\varphi_0}{\partial\omega_1\partial\omega_2} - 2\omega_1\omega_2^3\omega_3\frac{\partial^2\varphi_0}{\partial\omega_2\partial\omega_3} \\ &+ \omega_1(1+6\omega_1)\frac{\partial\varphi_0}{\partial\omega_1} + 2\omega_1\omega_2^3\frac{\partial\varphi_0}{\partial\omega_2} + 2\omega_1\omega_2^2\omega_3^3\frac{\partial\varphi_0}{\partial\omega_3} + \frac{c_2}{2c_1}\varphi_0 - a\varphi_0^{1-2c_1/c_2} = 0 \end{aligned} \tag{11}$$

$$\begin{aligned} 4\omega_1^3\frac{\partial^2\varphi_1}{\partial\omega_1^2} &+ \omega_1\omega_2^2(1+\omega_2^2)\frac{\partial^2\varphi_1}{\partial\omega_2^2} + \omega_1\omega_2^2\omega_3^2(1+\omega_3^2)\frac{\partial^2\varphi_1}{\partial\omega_3^2} - 4\omega_1^2\omega_2\frac{\partial^2\varphi_1}{\partial\omega_1\partial\omega_2} - 2\omega_1\omega_2^3\omega_3\frac{\partial^2\varphi_1}{\partial\omega_2\partial\omega_3} \\ &+ \omega_1(1+6\omega_1)\frac{\partial\varphi_1}{\partial\omega_1} + 2\omega_1\omega_2^3\frac{\partial\varphi_1}{\partial\omega_2} + \frac{k_2}{2c_1}\varphi_1 - an\varphi_0^{-2c_1/c_2}\varphi_1 - c\varphi_0^{(k_2-2c_1)/c_2} = 0. \end{aligned} \tag{12}$$

We first have to solve φ_0 from the nonlinear equation (11) and then the linear equation (12) for φ_1. If we consider φ_0 as a function of ω_1, ω_2, or ω_3 we obtain equations of the form

$$\frac{\mathrm{d}^2\varphi_0}{\mathrm{d}\omega_j} + h_1(\omega_j)\frac{\mathrm{d}\varphi_0}{\mathrm{d}\omega_j} + h_2(\omega_j)\varphi_0 + h_3(\omega_j)\varphi_0^n = 0. \tag{13}$$

This equation was studied by Euler *et al* (1989) and Duarte *et al* (1991). General conditions on the functions h_1, h_2 and h_3 were constructed so that (13) could be transformed, by an invertible point transformation, to the integrable equation $\mathrm{d}^2X/\mathrm{d}T^2 + X^n = 0$. The Painlevé test and the existence of a Lie point symmetry was also studied. By using the results from these articles we found that the ordinary differential equations which follow from (11) cannot be transformed to an integrable second-order equation: they have no Lie point symmetries, and they do not pass the Painlevé test for any $n \geqslant 2$. For more information on the Painlevé test and invertible-point transformations we refer the reader to Steeb and Euler (1988) and Steeb (1993). Note that there is a close correspondence between the existence of Lie symmetries and the Painlevé property (Euler *et al* 1993).

In order to find exact solutions for φ_0 from (11) we consider the ansatz

$$\varphi_0(\omega_j) = \omega_j^\beta \psi(z(\omega_j)) \qquad \frac{dz(\omega_j)}{d\omega_j} = \omega_j^\alpha \tag{14}$$

where α and β are real constants that must be determined. The following first-order approximate solutions for u are obtained:

(i) Let $\varphi_0 = \varphi_0(\omega_1)$ and $\varphi_1 = \varphi_1(\omega_1)$. It follows that

$$\begin{aligned} \varphi_0(\omega_1) &= a^{1/(1-n)} \frac{2(n+1)}{(1-n)^2} \omega_1^{1/(n-1)} \\ \varphi_1(\omega_1) &= \frac{c}{10} \omega^{-1} \exp\left(\frac{1}{2\omega_1}\right) \end{aligned} \tag{15}$$

where $k_2 = 2c_1$, $n \neq -1$, and

$$\frac{2(n+1)}{(1-n)^2} = \left(\frac{12}{n}\right)^{1/(n-1)}.$$

A first-order approximate solution $u = u_0 + \varepsilon u_1$, for the equation

$$\frac{\partial u}{\partial x_0} + \Delta u = a u^n + \varepsilon \left(\frac{a k_1 k_2}{c_2(c_2 - k_2)} u^n + \frac{a k_3 n}{k_2} u^{n-1} + c\right) \tag{16}$$

follows from (10) and (15).

(ii) Let $\varphi_0 = \varphi_0(\omega_2)$ and $\varphi_1 = \varphi_1(\omega_2)$. It follows that

$$\begin{aligned} \varphi_0(\omega_2) &= \frac{3}{a} \omega_2^2 \\ \varphi_1(\omega_2) &= \tilde{c}(-\tfrac{1}{3}\omega_2^{-1} + \omega_2 + \omega_2^2 \tan^{-1}\omega_2) + \frac{27c}{14a^3}(\omega_2^4 - \omega_2^2 \ln(1+\omega_2^2)) \end{aligned} \tag{17}$$

where $n = 2$, $\omega_1 = 1$, $k_2 = -4c_1$, and $\tilde{c} \in \mathcal{R}$. A first-order approximate solution, $u = u_0 + \varepsilon u_1$, for the equation

$$\frac{\partial u}{\partial x_0} + \Delta u = a u^2 + \varepsilon \left(-\frac{a k_1}{c_2 - k_2} u^2 + \frac{2 a k_3}{k_2} u + c u^3\right) \tag{18}$$

follows from (10) and (17).

3. Approximate in terms of u_0

We now consider the problem of finding a first-order approximate solution with the representation

$$u = u_0 + \varepsilon g(u_0) \tag{19}$$

where g is an arbitrary smooth function. System (3) now takes the form

$$\frac{\partial u_0}{\partial x_0} + \Delta u_0 = a u_0^n \tag{20}$$

$$\sum_{i=1}^{3} \left(\frac{\partial u_0}{\partial x_i}\right)^2 \frac{\mathrm{d}^2 g}{\mathrm{d}u_0^2} + a u_0^n \frac{\mathrm{d}g}{\mathrm{d}u_0} = a n u_0^{n-1} g(u_0) + f(u_0). \tag{21}$$

From the splitting condition of (21) we introduce a function $A(u_0)$ together with the compatibility problem

$$\sum_{i=1}^{3} \left(\frac{\partial u_0}{\partial x_i}\right)^2 = A(u_0) \qquad \frac{\partial u_0}{\partial x_0} + \Delta u_0 = a u_0^n. \tag{22}$$

In order to find compatible solutions for system (22) we investigate the symmetry of system (22).

Theorem 2. The infinitesimal functions ξ_j and η, in the Lie symmetry generator

$$Z = \sum_{j=0}^{3} \xi_j(x_0, \ldots, x_3, u) \frac{\partial}{\partial x_j} + \eta(x_0, \ldots, x_3, u) \frac{\partial}{\partial u}$$

are given by the following four cases:

(i) For n arbitrary ($n \neq 1$) and $A(u_0) = u_0^{n+1}$ it follows that

$$\xi_0 = 2c_{00}x_0 + d_0 \qquad \xi_1 = c_{00}x_1 + c_{12}x_2 + c_{13}x_3 + d_1$$

$$\xi_2 = c_{00}x_2 - c_{12}x_1 + c_{23}x_3 + d_2 \qquad \xi_3 = c_{00}x_3 - c_{13}x_1 - c_{23}x_2 + d_3$$

$$\eta = \frac{2c_{00}}{1-n} u_0.$$

(ii) For an arbitrary real constant n and arbitrary function $A(u_0)$ it follows that

$$\xi_0 = d_0 \qquad \xi_1 = c_{12}x_2 + c_{13}x_3 + d_1$$

$$\xi_2 = -c_{12}x_1 + c_{23}x_3 + d_2 \qquad \xi_3 = -c_{13}x_1 - c_{23}x_2 + d_3$$

$$\eta = 0.$$

(iii) For $n = 1$, $A(u_0) = 0$ and $a \neq 0$ it follows that

$$\xi_0 = 2c_{00}x_0 + d_0 \qquad \xi_1 = c_{00}x_1 + c_{12}x_2 + c_{13}x_3 + d_1$$

$$\xi_2 = c_{00}x_2 - c_{12}x_1 + c_{23}x_3 + d_2 \qquad \xi_3 = c_{00}x_3 - c_{13}x_1 - c_{23}x_2 + d_3$$

$$\eta = (2ac_{00}x_0 + b)u_0.$$

(iv) For $n = 1$ and $A(u_0) = u_0^2$ it follows that

$$\xi_0 = d_0 \qquad \xi_1 = c_{12}x_2 + c_{13}x_3 + d_1$$

$$\xi_2 = -c_{12}x_1 + c_{23}x_3 + d_2 \qquad \xi_3 = -c_{13}x_1 - c_{23}x_2 + d_3$$

$$\eta = bu_0.$$

Here b, c_{ij}, $d_j \in \mathcal{R}$ $(i, j = 0, \ldots, 3)$. The proof follows from the invariance conditions.

As an example of constructing compatible solutions for system (22) we consider the linear combination of the rotation in x_1 and x_2 with the translation in x_3, i.e.

$$x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2} + \alpha\frac{\partial}{\partial x_3}$$

where $\alpha \in \mathcal{R}$. This symmetry is valid for arbitrary n and arbitrary function $A(u_0)$. By solving the associated Lagrange system we obtain the symmetry ansatz

$$\begin{aligned} &u_0 = \varphi(\omega_1, \omega_2, \omega_3) \\ &\omega_1 = x_0 \qquad \omega_2 = x_1^2 + x_2^2 \qquad \omega_3 = x_3 + \alpha \sin^{-1}\frac{x_2}{\sqrt{x_1^2 + x_2^2}}. \end{aligned} \tag{23}$$

By using (23), system (22) reduces to

$$\begin{aligned} &\frac{\partial\varphi}{\partial\omega_1} + 4\omega_2\frac{\partial^2\varphi}{\partial\omega_2^2} + 4\frac{\partial\varphi}{\partial\omega_2} + \frac{\partial^2\varphi}{\partial\omega_3^2}\left(\frac{\alpha^2}{\omega_2} + 1\right) - a\varphi^n = 0 \\ &4\omega_2\left(\frac{\partial\varphi}{\partial\omega_2}\right)^2 + \alpha\left(\frac{\partial\varphi}{\partial\omega_3}\right)^2 - \omega_2 A(\varphi) = 0. \end{aligned} \tag{24}$$

Let us consider $\varphi = \varphi(\omega_2)$. System (24) reduces to

$$4\omega_2\frac{d^2\varphi}{d\omega^2} + 4\frac{d\varphi}{d\omega_2} - a\varphi^n = 0 \tag{25}$$

$$4\left(\frac{d\varphi}{d\omega_2}\right)^2 - A(\varphi) = 0. \tag{26}$$

With ansatz (14), where $\alpha = -1$ and $\beta = 1/(1-n)$, equation (25) takes the following form:

$$\frac{d^2\psi}{dz^2} - \frac{2}{n-1}\frac{d\psi}{dz} + \frac{1}{(n-1)^2}\psi + \frac{a}{4}\psi^n = 0.$$

This equation has no Lie symmetries, no point transformation to an integrable equation exists, nor does it pass the Painlevé test for any $n \geqslant 2$. A constant solution for arbitrary n is given by

$$\psi = \left(\frac{4}{a(n-1)^2}\right)^{1/(n-1)}.$$

From (14) together with (26) we obtain

$$\varphi = \left(\frac{4}{a(n-1)^2\omega_2}\right)^{1/(n-1)} \qquad A(\varphi) = \frac{a^2}{4}(n-1)^2\varphi^{2n}. \tag{27}$$

A solution of (22) then follows from (23) and (27). Using (27), equation (21) reduces to the linear equation

$$\frac{a}{4}(n-1)^2 u_0^n \frac{d^2 g}{du_0^2} + \frac{dg}{du_0} - n u_0^{-1} g = \frac{1}{a u_0^n} f(u_0). \tag{28}$$

First approximate solutions of (1) can now be constructed by solving (28) for a given function f. For example

$$u = \left(\frac{4}{a(n-1)^2(x_1^2+x_2^2)}\right)^{1/(n-1)} + \varepsilon \cos\left(\frac{4}{a(n-1)^2(x_1^2+x_2^2)}\right)^{1/(n-1)} \tag{29}$$

is a first-order approximate solution for the equation

$$\frac{\partial u}{\partial x_0} + \Delta u = au^n + \varepsilon\left(-\frac{a^2}{4}(n-1)^2 u^{2n}\cos u - nu^{-1}\cos u - \sin u\right). \tag{30}$$

Another example of a compatible solution for system (22) is obtained by considering the space translation symmetries which provides the ansatz

$$u_0 = \varphi(\omega) \qquad \omega = \boldsymbol{\alpha}\cdot\boldsymbol{x} := \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 \tag{31}$$

with α_i $(i = 1, \ldots, 3)$ arbitrary constants and $\boldsymbol{\alpha}^2 \neq 0$. System (22) now takes the form

$$\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\omega}\right)^2 = \frac{1}{\boldsymbol{\alpha}^2}A(\varphi) \qquad \frac{\mathrm{d}^2\varphi}{\mathrm{d}\omega^2} = \frac{a}{\boldsymbol{\alpha}^2}\varphi^n. \tag{32}$$

Let us consider $n = 0$ in (1) for the nonlinear functions given by theorem 1. For the function (7) it follows that

$$A(\varphi) = 2a\varphi \qquad u_0 = \frac{a}{2\boldsymbol{\alpha}^2}(\boldsymbol{\alpha}\cdot\boldsymbol{x})^2 \tag{33}$$

is a solution of system (32), so that

$$g(u_0) = \frac{k_1}{c_2 - k_2}u_0 + \frac{cc_2^2}{ak_2(2k_2 - c_2)}u_0^{k_2/c_2} + 2\tilde{c}_1 u_0^{1/2} + \tilde{c}_2.$$

Here $\tilde{c}_1$ and $\tilde{c}_2$ are arbitrary real constants. Thus a first-order approximate solution for the equation

$$\frac{\partial u}{\partial x_0} + \Delta u = a + \varepsilon\left(\frac{ak_1}{c_2 - k_2} + cu^{-(1-k_2/c_2)}\right) \tag{34}$$

is given by

$$u(x_0, x_1, x_2, x_3) = u_0 + \varepsilon\left(\frac{k_1}{c_2 - k_2}u_0 + \frac{cc_2^2}{ak_2(2k_2 - c_2)}u_0^{k_2/c_2} + 2\tilde{c}_1 u_0^{1/2} + \tilde{c}_2\right) \tag{35}$$

where u_0 is given by (33). In the same way as above, whereby we consider (21) together with (22) and the function (8), we obtain a first-order approximate solution for the equation

$$\frac{\partial u}{\partial x_0} + \Delta u = a + \varepsilon\left(\frac{ak_1}{c_2} + cu^{-1}\right)$$

as

$$u(x_0, x_1, x_2, x_3) = u_0 + \varepsilon\left(\frac{k_1}{c_2}u_0 + 2\tilde{c}_3 u_0^{1/2} - \frac{c}{a}\ln u_0 + \tilde{c}_4\right).$$

Here $\tilde{c}_3$, $\tilde{c}_4$ are arbitrary constants and u_0 is given by (33). Note that $n = 0$ is not valid for the function (9).

4. An approximate Lie–Bäcklund generator

By considering Lie–Bäcklund generators for system (3), i.e. first-order approximate Lie–Bäcklund generators for (1), we can construct approximate solutions for (1). Lie–Bäcklund generators for system (3) exist only in the one-space dimensional case with $a = 0$ and $f(u_0) = u_0^2$. Such a Lie–Bäcklund generator is given by

$$Z_{\mathrm{B}} = \frac{\partial^3 u_0}{\partial x_1^3}\frac{\partial}{\partial u_0} + \left(\frac{\partial^3 u_1}{\partial x_1^3} - 3u_0\frac{\partial u_0}{\partial x_1}\right)\frac{\partial}{\partial u_1}. \tag{36}$$

Note that a hierarchy of Lie–Bäcklund generators can be constructed from the recursion operator (Euler and Steeb 1992). From the linear combination of the Lie–Bäcklund generator and time translation, a first-order approximate solution $u = u_0 + \epsilon u_1$ for (1) follows, where u_0 and u_1 are given by

$$u_0 = \frac{\tilde{c}_4}{\tilde{c}_1^2}\exp(-\tilde{c}_1(\tilde{c}_1 x_0 + x_1)) + \tilde{c}_2 x_1 + \tilde{c}_3$$

$$\begin{aligned} u_1 = {}& \exp(-\tilde{c}_1(\tilde{c}_1 x_0 + x_1))\left(\frac{\tilde{c}_4^2}{2\tilde{c}_1^6}\exp(-\tilde{c}_1(\tilde{c}_1 x_0 + x_1)) - \frac{\tilde{c}_2\tilde{c}_4}{2\tilde{c}_1^3}x_1^2 + \left(\frac{\tilde{c}_2\tilde{c}_4}{\tilde{c}_1^4} - \frac{\tilde{c}_3\tilde{c}_4}{\tilde{c}_1^3}\right)x_1\right) \\ & + \frac{\tilde{c}_2^2}{12}x_1^4 + \frac{1}{6}\left(2\tilde{c}_2\tilde{c}_3 + \frac{\tilde{c}_2^2}{\tilde{c}_1}\right)x_1^3 + \frac{1}{2}\left(\tilde{c}_3^2 + \frac{\tilde{c}_2\tilde{c}_3}{\tilde{c}_1} - \frac{\tilde{c}_2^2}{\tilde{c}_1^2}\right)x_1^2 \\ & + \left(-\frac{\tilde{c}_2^2 x_0}{\tilde{c}_1} + \tilde{c}_5\right)x_1 + \left(-\frac{\tilde{c}_2\tilde{c}_3}{\tilde{c}_1} + \frac{\tilde{c}_2^2}{\tilde{c}_1^2}\right)x_0 + \tilde{c}_6. \end{aligned}$$

Here $\tilde{c}_1, \ldots, \tilde{c}_6 \in \mathcal{R}$.

5. On the method of Baikov *et al* (1989)

Finally, we demonstrate that the approximate symmetry approach of Baikov *et al* (1989) is not effective for constructing solutions for equation (1). In this approach a first approximate symmetry is represented in the form

$$Z = \sum_{j=0}^{3}(\xi_j + \varepsilon\tilde{\xi}_j)\frac{\partial}{\partial x_j} + (\eta + \varepsilon\tilde{\eta})\frac{\partial}{\partial u}. \tag{37}$$

The invariance of a partial differential equation with parameter ε is then considered whereupon the determining equations for the infinitesimal functions $\xi_j, \tilde{\xi}_j, \eta, \tilde{\eta}$ are obtained by equating to zero the coefficients of zero and the first power of ε. For equation (1), with the function f given by

$$f(u) = a(1-n)\frac{\tilde{b}_1(1-n) - 2\tilde{c}_{00}}{2c_{00}}u^n \ln u + an\tilde{b}_2\frac{1-n}{2c_{00}}u^{n-1} + cu^n \tag{38}$$

($c_{00} \neq 0, n \neq 0, n \neq 1$), the infinitesimal functions in (37) are as follows

$$\xi_0 = 2c_{00}x_0 + d_0 \qquad \tilde{\xi}_0 = 2\tilde{c}_{00}x_0 + \tilde{d}_0$$

$$\xi_1 = c_{00}x_1 + c_{12}x_2 + c_{13}x_3 + d_1 \qquad \tilde{\xi}_1 = \tilde{c}_{00}x_1 + \tilde{c}_{12}x_2 + \tilde{c}_{13}x_3 + \tilde{d}_1$$

$$\xi_2 = -c_{12}x_1 + c_{00}x_2 + c_{23}x_3 + d_2 \qquad \tilde{\xi}_2 = -\tilde{c}_{12}x_1 + \tilde{c}_{00}x_2 + \tilde{c}_{23}x_3 + \tilde{d}_2$$

$$\xi_3 = -c_{13}x_1 - c_{23}x_2 + c_{00}x_3 + d_3 \qquad \tilde{\xi}_3 = -\tilde{c}_{13}x_1 - \tilde{c}_{23}x_2 + \tilde{c}_{00}x_3 + \tilde{d}_3$$

$$\eta = \frac{2c_{00}}{1-n}u \qquad \tilde{\eta} = \tilde{b}_1 u + \tilde{b}_2$$

where c_{ij}, d_j, $\tilde{c}_{ij}$, $\tilde{d}_j$, and $\tilde{b}_k$ ($i, j = 0, \dots 3,\ k = 1, 2$) are arbitrary real constants. Let us consider the scaling symmetries

$$D = 2x_0\frac{\partial}{\partial x_0} + x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3} + \frac{2}{1-n}u\frac{\partial}{\partial u}$$

i.e. $c_{00} = 1, c_{ij} = 0, b_k = 0$ ($i \neq j = 1, \dots, 3,\ k = 1, 2$), and

$$\tilde{D} = \varepsilon u\frac{\partial}{\partial u}$$

i.e. $c_{ij} = 0, b_1 = 1, b_2 = 0$ ($i, j = 0, \dots, 3$). By solving the Lagrange system for the combination $D + \tilde{D}$ we obtain the symmetry ansatz

$$u = x_0^{[2+\varepsilon(1-n)]/[2(1-n)]}\varphi(\omega_1, \omega_2, \omega_3)$$

$$\omega_1 = \frac{x_0}{x_1^2} \qquad \omega_2 = \frac{x_1}{x_2} \qquad \omega_3 = \frac{x_2}{x_3}.$$

Equation (1), together with function (38), then reduces to

$$4\omega_1^3\frac{\partial^2\varphi}{\partial\omega_1^2} + \omega_1\omega_2^2(1+\omega_2^2)\frac{\partial^2\varphi}{\partial\omega_2^2} + \omega_1\omega_2^2\omega_3^2(1+\omega_3^2)\frac{\partial^2\varphi}{\partial\omega_3^2} - 4\omega_1^2\omega_2\frac{\partial^2\varphi}{\partial\omega_1\partial\omega_2} - 2\omega_1\omega_2^3\omega_3\frac{\partial^2\varphi}{\partial\omega_2\partial\omega_3}$$
$$+ \omega_1(1+6\omega_1)\frac{\partial\varphi}{\partial\omega_1} + 2\omega_1\omega_2^3\frac{\partial\varphi}{\partial\omega_2} + 2\omega_1\omega_2^2\omega_3^3\frac{\partial\varphi}{\partial\omega_3} + \frac{2+\varepsilon(1-n)}{2(1-n)}\varphi_0$$
$$-ax_0^{\varepsilon(n-1)/2}\varphi^n - \varepsilon x_0^{\varepsilon(n-1)/2}\left[\frac{a(1-n)^2}{2}\varphi^n \ln\left(x_0^{[2+\varepsilon(1-n)]/[2(1-n)]}\varphi\right) + c\varphi^n\right] = 0. \tag{39}$$

From (39) it is clear that solutions of φ in terms of ω_1, ω_2 and ω_3 can only be constructed if $\varepsilon = 0$.

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